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ABSTRACT

A convex set in a vector space is a set of points such that whenever x_1, x_2 belong to the set, then all points of the form $\lambda x_1 + (1-\lambda)x_2$, where λ is in the interval $[0, 1]$, also belong to the set.

The discussion that follows deals with a certain type of function which has a convex domain. In particular, we consider convex functions whose domains are closed, bounded intervals of real numbers.

In addition to defining a "convex function," properties of convexity and conditions for convexity are established. These properties and conditions are then used to establish necessary and sufficient conditions for convexity.

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1. Properties of Convex Functions

We make the following definitions relative to convex functions:

<u>Definition 1.1</u> A real valued function f defined on the closed, bounded interval, $[a, b]$, of the reals is said to be <u>convex</u> (or <u>concave up</u>), if

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) \text{ for all}$$
$$\lambda \in [0, 1] \text{ and all } x_1, x_2 \in [a, b].$$

Graphically, the function f is convex if the portion of its graph in every subinterval of its domain lies on or below its secant line. See Figure 1-1.

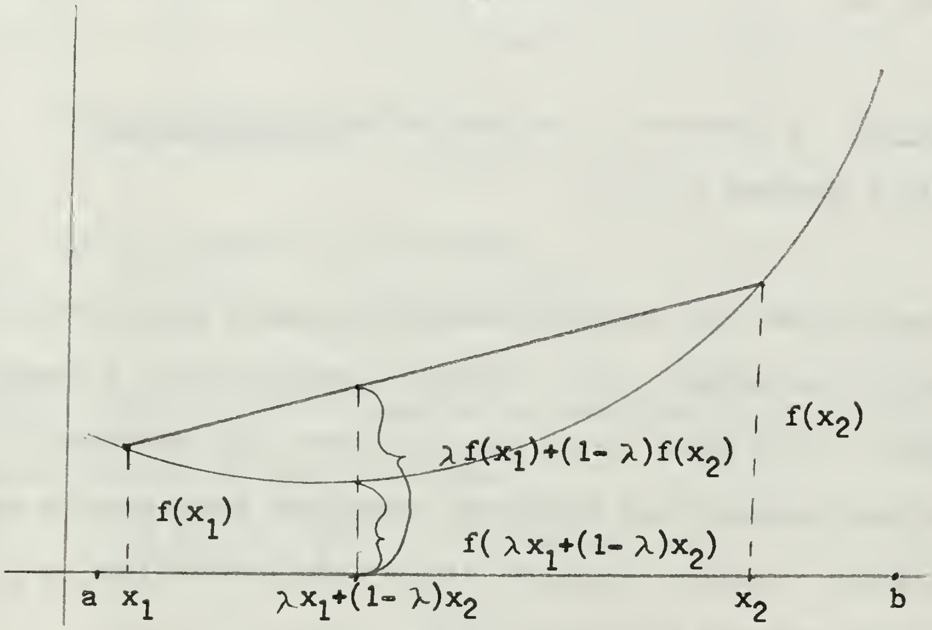

Figure 1-1

Comments. It is sufficient to assume that the domain of the function is a closed, bounded interval $[a, b]$, since a function is convex on any interval I if, and only if, it is convex on all closed, bounded subintervals of I.

Definition 1.2 A real valued function defined on a closed bounded interval $[a, b]$ is said to be *concave* (or *concave down*), if $f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1 + (1-\lambda)f(x_2)$ for all $\lambda \in [0, 1]$ and all $x_1, x_2 \in [a, b]$.

Since the function (f) is concave up if and only if (-f) is concave down, the discussion can be limited to functions which are concave up.

Definition 1.3 A function f is said to be *non-decreasing* if $f(x_1) \geq f(x_2)$ whenever $x_1 \geq x_2$.

A function that has the basic convexity property described in Definition 1.1 has certain other fundamental properties as a result of convexity. It is these latter properties that are considered first, so that necessary and sufficient conditions for convexity can be established. Consider initially, linear functions defined on a closed bounded interval $[a, b]$.

Definition 1.4 A function f is *linear* if it is of the form $f(x) = Ax + B$ where A and B are constants.

Theorem 1.1 If f is a linear function then:

$f(a+h) = \frac{h}{k} f(a+k) + \frac{k-h}{k} f(a)$. See Figure 1-2.

Whenever $k \geq h > 0$.

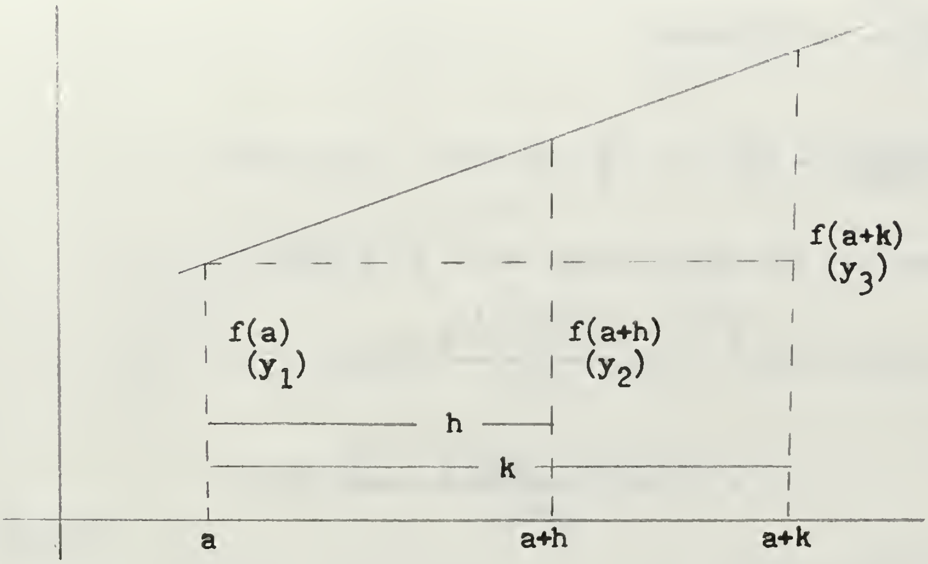

Figure 1-2

In usual notation this means

$$y_2 = \frac{h}{k} y_3 + \frac{h-k}{k} y_1$$

Proof: Using similar triangles, we see that $\frac{y_2-y_1}{y_3-y_1} = \frac{h}{k}$; by cross multiplying and simplifying

$$y_2 = \frac{h}{k} y_3 + \left(\frac{k-h}{k}\right) y_1 \quad \text{which means}$$

$$f(a+h) = \frac{h}{k} f(a+k) + \left(\frac{k-h}{k}\right) f(a).$$

Corollary 1.1.1 If f is a linear function then f is convex. I.e., Theorem 1.1 implies the convexity property.

Proof:

Take $a = x_1$ and $a + k = x_2$ then $k = x_2 - x_1$.

Define $h = (\lambda - 1)\, x_1 + (1 - \lambda) x_2$.

Solving for λ

$$\lambda = \frac{x_2-(x_1+h)}{x_2-x_1} \text{ which means}$$

$$\lambda = \frac{(a+k)-(a+h)}{(a+k)-a} = \frac{k-h}{k} = 1 - \frac{h}{k}, \text{ so that } \lambda \in [0, 1].$$

Using Theorem 1.1 and substituting for a, h, k gives

$$f(x_1 + \lambda x_1 - x_1 + (1-\lambda)x_2 = \frac{(\lambda - 1)x_1 + (1-\lambda)x_2}{x_2-x_1} f(x_2)$$

$$+ \frac{x_2-x_1-(\lambda-1)x_1-(1-\lambda)x_2}{x_2-x_1} f(x_1)$$

Simplifying this gives

$f(\lambda x_1 + (1-\lambda)x_2) = \lambda f(x_1) + (1-\lambda)f(x_2)$ which implies the convexity property.

Comment. It is clear that linear functions also satisfy the concavity property. Thus, linear functions are both convex and concave. They are the only such functions.

Now we consider general convex functions and their resultant properties.

Theorem 1.2 If f is convex and $x_1 < x_2 < x_3$ then:

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2}.$$

I.e., as we move to the right, the slope of the secant line increases. See Figure 1-3.

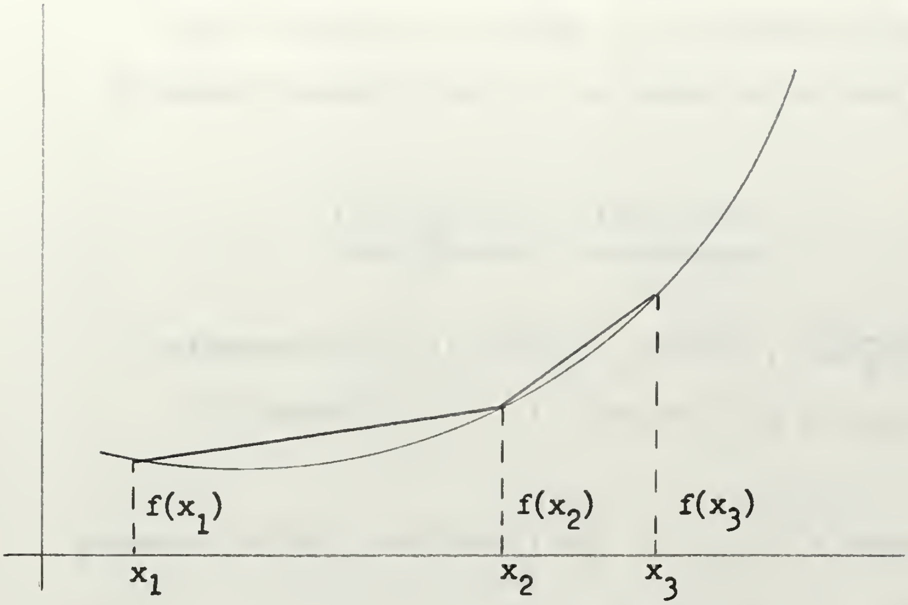

Figure 1-3

<u>Proof</u>:

Let $\lambda = \frac{x_3-x_2}{x_3-x_1}$ (Note that $0 < \lambda < 1$), so that

$x_2 = \lambda x_1 + (1-\lambda)x_3$. Since f is convex,

$f(x_2) = f(\lambda x_1 + (1-\lambda)x_3) \leq \lambda f(x_1) + (1-\lambda)f(x_3)$.

Substituting for λ and simplifying gives

$$f(x_2) \leq \left(\frac{x_3-x_2}{x_3-x_1}\right) f(x_1) + \left(\frac{x_2-x_1}{x_3-x_1}\right) f(x_3).$$

Since (x_3-x_1) is positive, it follows that

$(x_3-x_1)f(x_2) \leq (x_3-x_2)f(x_1) + (x_2-x_1)f(x_3)$.

Expanding and then subtracting $x_2f(x_2)$ from both sides gives

$(x_3-x_2)(f(x_2) - f(x_1)) \leq (x_2-x_1)(f(x_3) - f(x_2))$.

Since (x_3-x_2) and (x_2-x_1) are positive, this means that

$$\frac{f(x_2) - f(x_1)}{x_2-x_1} \leq \frac{f(x_3) - f(x_2)}{x_3-x_2}.$$

<u>Theorem 1.3</u> If f is convex and $0 \neq h_2 \geq h_1 \neq 0$, then for each t in the domain of f and h_1, h_2 such that $(t+h_1)$ and $(t+h_2)$ are in the domain of f, the following inequality holds:

$$\frac{f(t+h_1)-f(t)}{h_1} \leq \frac{f(t+h_2)-f(t)}{h_2} \quad .$$

I.e., $\frac{f(t+h)-f(t)}{h}$, denoted $\Delta_h f(t)$, is a non-decreasing function <u>of h</u>, $(h \neq 0)$ for each t. (C.f. Theorem 1.4.)

<u>Proof</u>: Assume $h_2 \geq h_1 > 0$. See Figure 1-4. By the convexity hypothesis,

$$f(t+h_1) \leq \frac{h_1}{h_2} f(t+h_2) + \frac{h_2-h_1}{h_2} f(t)$$

where $0 < \frac{h_1}{h_2} \leq 1$; $0 \leq \frac{h_2-h_1}{h_2} < 1$; $\frac{h_2-h_1}{h_2} + \frac{h_1}{h_2} = 1$.

Since h_2 is positive it follows that

$$h_2 f(t+h_1) \leq h_1 f(t+h_2) + h_2 f(t) - h_1 f(t).$$

By transposing and factoring, this gives

$$h_2 \left[f(t+h_1) - f(t)\right] \leq h_1 \left[f(t+h_2) - f(t)\right]$$

Since h_1, h_2 are positive, this means

$$\frac{f(t+h_1)-f(t)}{h_1} \leq \frac{f(t+h_2)-f(t)}{h_2}$$

Thus $\Delta_h f(t) = \frac{f(t+h)-f(t)}{h}$ is a non-decreasing function of h for $h > 0$.

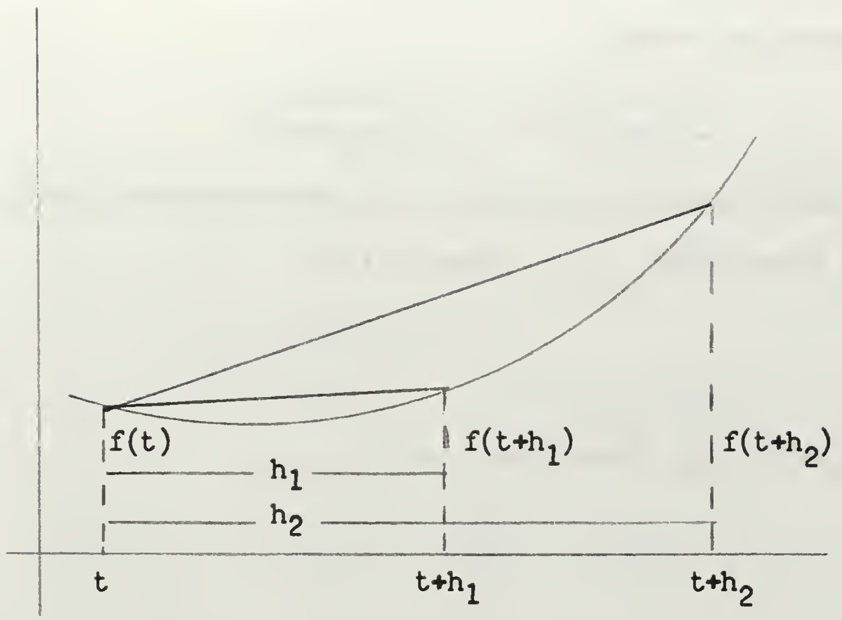

Figure 1-4

Similarly it can be shown that when $h_1 \leq h_2 < 0$, then

$$\frac{f(t+h_1)-f(t)}{h_1} \leq \frac{f(t+h_2)-f(t)}{h_2}$$

which means that $\triangle_h f(t)$ is a non-decreasing function of h for $h < 0$.

If $h_1 < 0 < h_2$, then by letting $\lambda = \frac{h_2}{h_2-h_1}$ and applying the convexity hypothesis, we again have

$$\frac{f(t+h_1)-f(t)}{h_1} \leq \frac{f(t+h_2)-f(t)}{h_2}$$

Combining these results, we see that $\triangle_h f(t)$ is a non-decreasing function of h ($h \neq 0$) for each t. Clearly, for $h = 0$, the function $\triangle_h f(t)$ is not defined.

Theorem 1.4 If f is convex and $h > 0$ and fixed, then for $t_1 \le t_2$ such that t_1, t_2, $t_1 + h$, $t_2 + h$ are all in the domain of f, the following inequality holds:

$$\frac{f(t_1+h)-f(t_1)}{h} \le \frac{f(t_2+h)-f(t_2)}{h}$$

I.e., $\frac{f(t+h)-f(t)}{h}$ denoted by $\Delta_h f(t)$ is a non-decreasing function of t for $h > 0$ and fixed. (C.F. Theorem 1.3)

Proof:

Case 1 $(t_1+h) < t_2$. (See Figure 1-5.)

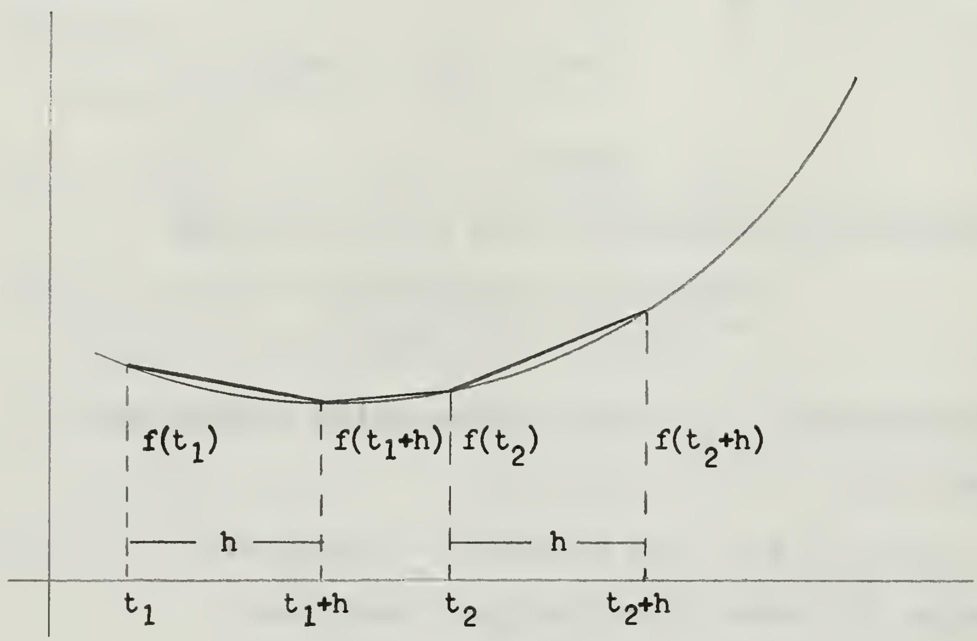

Figure 1-5

From two applications of Theorem 1.2,

$$\frac{f(t_1+h)-f(t_1)}{h} \le \frac{f(t_2)-f(t_1+h)}{t_2-(t_1+h)} \le \frac{f(t_2+h)-f(t_2)}{h}$$

Case 2 $(t_1+h) > t_2$. See Figure 1-6.

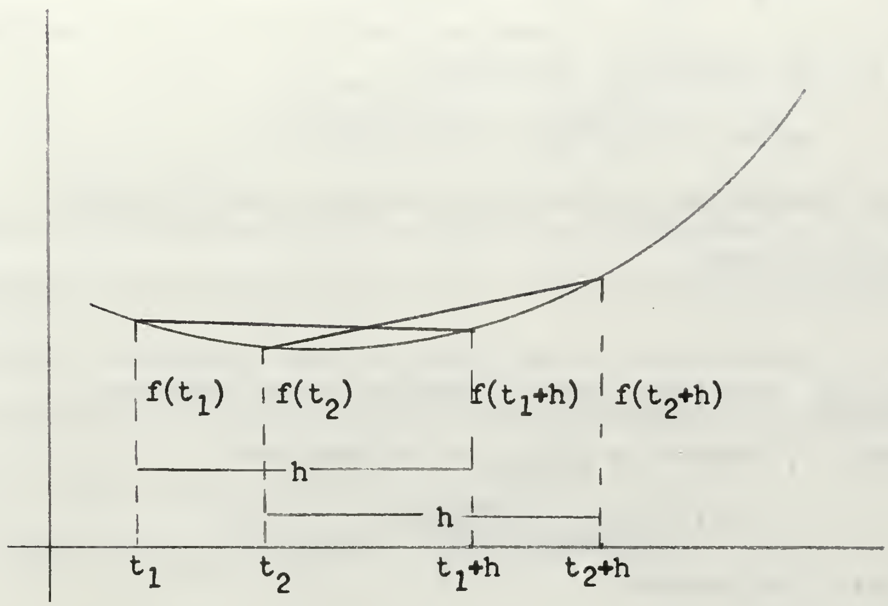

Figure 1-6

Again by two applications of Theorem 1.2,

$$\frac{f(t_1+h)-f(t_1)}{h} \leq \frac{f(t_1+h)-f(t_2)}{(t_1+h)-t_2} \leq \frac{f(t_2+h)-f(t_2)}{h}$$

The special case where $t_2 = t_1 + h$ is shown directly from Theorem 1.2. Similarly, if $h < 0$, the same inequality holds. In [5] the following is shown.

Theorem 1.5 If f is convex and $f(0) = 0$, then $\frac{f(mt)}{m}$ is a non-decreasing function of m for $m > 0$, for each t, whenever t and mt are in the domain of f.

I.e., $m_2 f(m_1 t) \geq m_1 f(m_2 t)$ where $m_1 \geq m_2 > 0$.

Proof: Consider the point 0 and any other point t in the domain of f. Take $m_1 \geq m_2 > 0$ such that $m_1 t$ and $m_2 t$ are in the domain of f. From the convexity property,

$$f(m_2t) \le \frac{m_2}{m_1} f(m_1t) - \frac{m_1-m_2}{m_1} f(0)$$

but $f(0) = 0$ by hypothesis. Therefore

$$m_2f(m_1t) \ge m_1f(m_2t) \text{ for } m_1 \ge m_2 > 0.$$

Before considering the remaining theorems, the following definitions are given:

Definition 1.5 The right-hand derivative of the function f at the point t, denoted by $D^+f(t)$, is defined as:

$$D^+f(t) = \lim_{h \to 0^+} \frac{f(t+h)-f(t)}{h}$$

whenever this limit exists.

Definition 1.6 The left-hand derivative of the function f at the point t, denoted by $D^-f(t)$, is defined as:

$$D^-f(t) = \lim_{h \to 0^-} \frac{f(t+h)-f(t)}{h}$$

whenever this limit exists.

Theorem 1.6 If f is convex on the open interval (a, b) then for every $t \in (a, b)$, $D^+f(t)$ and $D^-f(t)$ exist and moreover

$$-\infty < D^-f(t) \le D^+f(t) < \infty .$$

Proof: Take any point $t \in (a, b)$ and consider $D^+f(t)$. By Theorem 1.3, $\Delta_h f(t) = \frac{f(t+h)-f(t)}{h}$ is a non-decreasing function of h for every t in the domain of f. Thus:

$$\lim_{h \to 0^+} \frac{f(t+h)-f(t)}{h} = \inf_{h > 0} \frac{f(t+h)-f(t)}{h} .$$

Using Definition 1.5, this implies $D^+f(t)$ exists and is finite. Since the interval is open, we can find a point t_1 such that $a < t_1 < t$. Then for every $h > 0$,

$$\frac{f(t+h)-f(t)}{h} \geq \frac{f(t)-f(t_1)}{t-t_1}$$

Since the right-hand side is finite and independent of h, the infinum must be greater than $-\infty$. Again since the interval is open, there does exist at least one $h > 0$ such that t and $t+h$ are both in (a, b), so that the infinum must be less than $+\infty$. In a similar manner it can be shown that

$$D^-f(t) = \lim_{h \to 0^-} \frac{f(t+h)-f(t)}{h} = \sup_{h < 0} \frac{f(t+h)-f(t)}{h}$$

Thus, $D^-f(t)$ also exists and is finite. Again by Theorem 1.3, $\Delta_h f(t)$ is a non-decreasing function of h for every $t \in (a, b)$ which implies that $D^-f(t) \leq D^+f(t)$ for every t. Since if $h > 0 > k$, we have that

$$\frac{f(t+h)-f(t)}{h} \geq \frac{f(t+k)-f(t)}{k} .$$

The left-hand side is independent of k and the right-hand side is independent of h. Thus $D^+f(t) \geq D^-f(t)$. Thus $-\infty < D^-f(t) \leq D^+f(t) < \infty$.

Comments: In Theorem 1.6, the function was taken to be convex on the open interval (a, b). There exist functions which are convex on the closed interval $[a, b]$ for which $D^+f(a) = -\infty$. Such a function is $f(x) = -\sqrt{x}$ defined on the interval $[0, 1]$. Similarly, $D^-f(b)$ may be $+\infty$. The function $f(x) = \sqrt[3]{x}$ defined on the closed interval $[-1, 0]$ is convex on the interval, but $D^-f(0) = +\infty$. Thus to ensure finiteness of $D^-f(t)$ and $D^+f(t)$, we must consider the open interval (a, b).

Theorem 1.7 If f is convex on the closed, bounded interval $[a, b]$ then f is continuous on the open interval (a, b). However, f need not be continuous on $[a, b]$.

Proof: f convex on $[a, b]$ implies that f is convex on (a, b). Let t be an arbitrary point in (a, b). Then, by Theorem 1.6, both $D^-f(t)$ and $D^+f(t)$ exist and are finite. Let $\epsilon > 0$ be arbitrary and small.

Define M as the maximum absolute value of $D^-f(t)$ and $D^+f(t)$. I.e.,

$$M = \max\left\{ \left| D^-f(t) \right|, \left| D^+f(t) \right| \right\}$$

Choose δ so small that $\delta < \frac{\epsilon}{M+1}$ and simultaneously

$$\left| \frac{f(t+h)-f(t)}{h} - D^+f(t) \right| < 1 \quad \text{if } \delta > h > 0,$$

$$\left| \frac{f(t+h)-f(t)}{h} - D^-f(t) \right| < 1 \quad \text{if } 0 > h > -\delta.$$ (In both cases $|h| < \delta$.)

Suppose $0 < h < \delta$. Then

$$\left| f(t+h)-f(t) \right| = \left| (h)\left(\frac{f(t+h)-f(t)}{h} \right) \right| = |h| \left| \frac{f(t+h)-f(t)}{h} \right| .$$

It follows that

$$\left| f(t+h)-f(t) \right| \leq |h| \left| \frac{f(t+h)-f(t)}{h} - D^+f(t) + D^+f(t) \right|$$

$$\leq |h| \left\{ \left| \frac{f(t+h)-f(t)}{h} - D^+f(t) \right| + \left| D^+f(t) \right| \right\}.$$

Since $|h| < \delta$; $\left| \frac{f(t+h)-f(t)}{h} - D^+f(t) \right| < 1$, and $\left| D^+f(t) \right| \leq M$

so that $|h| \left\{ \left| \frac{f(t+h)-f(t)}{h} - D^+f(t) \right| + \left| D^+f(t) \right| \right\} < \delta \ (M+1) < \epsilon$.

Hence, $\left| f(t+h) - f(t) \right| < \epsilon$. Similarly, for $0 > h > -\delta$, $\left| f(t+h) - f(t) \right| < \epsilon$. Since ϵ was arbitrary f is continuous.

Comments: The function $f(x) = \begin{cases} x^2 \text{ when } -1 < x < 1 \\ 2 \text{ if } x = 1 \text{ or } x = -1. \end{cases}$ is an example of a function that is convex on a closed interval, namely, $[-1, 1]$ but is discontinuous at the end-points. Thus, convexity on a closed, bounded interval $[a, b]$ does not imply continuity on that interval.

There exist many non-convex functions which are continuous, for example: $f(x) = x^3$ defined on the interval $[-1, 1]$, so that, obviously, continuity does not imply convexity.

2. m-Convexity

Definition 2.1 Suppose that $m \in [0, 1]$. A function is called m-convex if, for every x_1, x_2 in the domain of f, we have $f(mx_1 + (1-m)x_2) \le mf(x_1) + (1-m)f(x_2)$. It follows that f is m-convex for all $m \in [0, 1]$, if, and only if, f is convex.

Boas [4], proves that if the function is 1/2-convex and continuous, then the function is convex. 1/2-convexity is often called "midpoint convexity."

In this section, it is shown that a milder hypothesis will suffice for convexity; namely, an m-convex function that is bounded on its domain is convex.

Definition 2.1 has m in the closed interval $[0, 1]$. It is clear that if $m = 0$ or $m = 1$, then every function is m-convex. Therefore, in determining sufficient conditions for convexity based on m-convexity, m is considered to be in the open interval $(0, 1)$.

Theorem 2.2 from [3] shows, by example, that an additional condition on the function, for example boundedness, is necessary for m-convexity to imply convexity.

Theorem 2.1: Let f be a function which is m-convex for some m in $(0, 1)$ and let f be bounded on its domain. Then f is convex everywhere on its domain. (The following proof is a generalization of the proof found in [1] for the case of midpoint convexity.)

Proof: Let $p = \frac{1}{m}$. Then $p > 1$. By assumption

$$f\left(\frac{x_1+(p-1)x_2}{p}\right) \leq \frac{1}{p}\left[f(x_1) + (p-1)f(x_2)\right]$$

for any x_1, x_2 in the domain of f. Suppose, to the contrary, that f is not convex. Then there exist x_1, x_2 and λ ($\lambda \in (0, 1)$) such that $f(\lambda x_1+(1-\lambda)x_2) - \lambda f(x_1) - (1-\lambda)f(x_2) = s > 0$. Without loss of generality we assume that $x_1 < x_2$. Clearly this $\lambda \neq \frac{1}{p}$ since this would contradict the m-convexity of the given function.

Since subtracting a linear function from f will not affect either convexity or boundedness on the closed interval $[x_1, x_2]$, we can assume that $f(x_1) = f(x_2) = 0$. Now we have $f(\lambda x_1 + (1-\lambda)x_2) = s > 0$.

Since $\lambda \neq \frac{1}{p}$, then $\lambda < \frac{1}{p}$ or $\lambda > \frac{1}{p}$. If $\lambda < \frac{1}{p}$, let $\lambda_1 = p\lambda$, and if $\lambda > \frac{1}{p}$, let $\lambda_1 = \frac{p\lambda - 1}{p-1}$.

Consider first the case where $\lambda < \frac{1}{p}$. Consider the two points $[\lambda_1 x_1 + (1-\lambda_1)x_2]$ and x_2. Since $0 < \lambda_1 < 1$, the two points are in $[x_1, x_2]$.

Applying the m-convexity inequality:

$$f\left(\frac{\lambda_1 x_1+(1-\lambda_1)x_2 + (p-1)x_2}{p}\right) \leq \frac{1}{p}\left[f(\lambda_1 x_1 + (1-\lambda_1)x_2) + (p-1)f(x_2)\right].$$

Substituting $\lambda_1 = p\lambda$ on the left gives

$$f\left(\frac{p\lambda x_1+(1-p\lambda)x_2+(p-1)x_2}{p}\right) - \frac{p-1}{p} f(x_2) \le \frac{1}{p} f(\lambda_1 x_1 + (1-\lambda_1)x_2).$$

Since $f(x_2) = 0$, by simplifying the left side we have

$f(\lambda x_1+(1-\lambda)x_2) \le \frac{1}{p} f(\lambda_1 x_1+(1-\lambda_1)x_2)$ which implies

$f(\lambda_1 x_1+(1-\lambda_1)x_2) > ps$.

Similarly, for the case where $\lambda > \frac{1}{p}$ consider the two points x_1 and $[\lambda_1 x_1+(1-\lambda_1)x_2]$. Since $0 < \lambda_1 < 1$, the two points are in $[x_1, x_2]$.

Applying the m-convexity inequality and simplifying, we can show that $f(\lambda_1 x_1+(1-\lambda_1)x_2) > \frac{p}{p-1} s$.

Let $q = \min(p, \frac{p}{p-1})$ which means $q > 1$. Then by repeating the above argument, we can find a sequence $\{\lambda_k\}$ such that $0 < \lambda_k < 1$ and such that

$$f(\lambda_k x_1 + (1-\lambda_k)x_2) > q^k s$$

Since $q > 1$, $q^k s \to \infty$ as $k \to \infty$, so that f is unbounded on the interval $[x_1, x_2]$, contradicting our hypothesis that f is bounded. Therefore, we have that if f is m-convex for some m in $(0, 1)$ and f is bounded on its domain, then f is convex.

<u>Comments</u>: By [2] it is proven that if the function f is m-convex and bounded on <u>some</u> non-trivial subinterval of its domain, it is bounded on every closed bounded subinterval of its domain. Using this fact, it is clear that Theorem 2.1 will hold if its statement is changed to read, "Let

f be a function which is m-convex for some m in (0, 1), and let f be bounded on some non-trivial subinterval of its domain. Then f is convex everywhere on its domain."

Theorem 2.2 Let $m \in (0, 1)$ be arbitrary. Then there exists functions which are m-convex, but are not convex. (See [3].)

Proof: For fixed m in (0, 1) let M denote the smallest field of real numbers which contains m. I.e., M is the set of all numbers expressible in the form

$$\frac{a_0 m^r + a_1 m^{r-1} + \dots + a_r}{b_0 m^s + b_1 m^{s-1} + \dots + b_s},$$

where r, s are non-negative integers, a_i, b_i are integers, and $b_0 m^s + b_1 m^{s-1} + \dots + b_s \neq 0$.

Consider R, the set of all reals, as a vector space over M.

Select a basis for this vector space, call it Y. (Ensure $1 \in Y$). Then Y is a set of reals, linearly independent over M. Moreover, every x is expressible by

(1) $x = \mu_1 y_1 + \dots + \mu_n y_n$ (distinct $y_k \in Y$ and $\mu_k \in M$).

This is unique except for zero terms. (Note that $M = My^* = \{\mu y^*: \mu \in M\}$ where $y^* \in Y \cap M = \{1\}$.)

We note the following properties: (See Appendix I)

(a) M is countable.

(b) M is everywhere dense in the reals.

(c) Y is not countable.

(d) If $y^* \in Y - M$ then the set $My^* = \{\mu y^* : \mu \in M\}$ is everywhere dense in the reals.

(e) $\mu_1 y_1 + \dots + \mu_n y_n = 0$ if, and only if, $\mu_1 = \dots = \mu_n = 0$. ($Y$ is linearly independent over M).

We will now construct a function f which is m-convex but not convex.

Given any real number x, define $f(x)$ as the coefficient μ_i (possibly zero) of 1 in (1). I.e., f is the projection mapping onto M, so that $f|M$ is the identity, $f|N = 0$ where M is the smallest subspace that contains 1, and N is the smallest subspace that contains $(Y-M)$.

A simple computation shows the defined function is m-linear and hence m-convex.

Consider each x in the everywhere dense set M, then $x = x \cdot 1$ and $f(x) = x$. Now if $y^* \in (Y-M)$, $f(x) = 0$ on the everywhere dense set My^*. From this we conclude that f is not continuous. (In fact, f is discontinuous everywhere. If m is rational, f is the familiar example of a function which is <u>additive</u> (i.e., $f(x_1+x_2) = f(x_1) + f(x_2)$ but not <u>homogeneous</u> (i.e., $f(kx) = kf(x)$).

We now show that f is not convex. I.e., for some x_1, x_2 and for some λ such that $0 < \lambda < 1$

$f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda)f(x_2)$. Suppose $y^* \in Y - M$. Since M is dense, there exists $m_0 \in M$ such that $m_0 y^* \in (0, 1)$. (Note that $y^* \neq 0$.) But $My^* = M(m_0 y^*)$

so that we can replace y^* by $m_0 y^*$ in the basis. (Note that this leaves the definition of f unchanged.)

Choose $y^* \in (Y - M)$ such that $0 < y^* < 1$. Take $\lambda = y^*$. If f is convex, we have for $x_1 = 0$ and $x_2 = 1$, $f\left[y^* \cdot 0 + (1-y^*)1\right] \leq y^*f(0) + (1-y^*)f(1)$. Reducing, this gives $f(1-y^*) \leq y^* \cdot 0 + (1-y^*) \cdot 1 = (1-y^*)$. 1 is in M and y^* is in $(Y-M)$; therefore, $f(1-y^*) = 1$. This gives us $1 \leq 1 - y^*$ or $y^* \leq 0$, but this is a contradiction since $y^* \in (0, 1)$.

From the above we see that for every $0 < m < 1$ there exists a function that is m-convex but not convex since there exists $0 < \lambda < 1$ for which λ-convexity does not hold.

3. Necessary and Sufficient Conditions for Convexity

Theorem 3.1: The function f is convex on the open interval (c, d) if, and only if, f is convex on every closed subinterval $[a, b] \subset (c, d)$.

Proof: It is clear that if f is convex on (c, d) then f is convex on every $[a, b] \subset (c, d)$.

Conversely, suppose that f is not convex on (c, d). Then there exist x_1, x_2 in (c, d) and λ in $(0, 1)$ such that $f(\lambda x_1 + (1 - \lambda)x_2) > \lambda f(x_1) + (1-\lambda)f(x_2)$without loss of generality take $x_1 < x_2$. But this implies that f is not convex on $[x_1, x_2] \subseteq (c, d)$.

Comments: Using an argument similar to that in the above theorem and the fact that by Theorem 1.7, convexity on $[a, b]$ implies continuity on (a, b), it can be shown that if f is convex on the open interval (c, d) then f is continuous on (c, d). It is noted that if f is convex on the open interval (c, d), then f need not be bounded. For example, $f(x) = \frac{1}{x}$ is convex on $(0, 1)$, yet f is not bounded. A function can be convex on every open subinterval of a bounded closed interval and yet not be convex on the closed interval. For example, let $f(x) = \begin{cases} x^2 & -1 < x < 1 \\ 0 & x = -1 \text{ or } x = 1. \end{cases}$

Theorem 3.2: Suppose f is m-convex and defined on a closed bounded interval. Then f is convex if, and only if, it is bounded on its domain.

Proof: If f is convex on $[a, b]$, it is bounded on $[a, b]$.

Suppose that $y \in [a, b]$. Then for some (unique) $\lambda \in [0, 1]$, $y = \lambda a + (1-\lambda)b$, so that

$$f(y) \leq \lambda f(a) + (1-\lambda)f(b) \leq \left|\lambda f(a) + (1-\lambda)f(b)\right|$$
$$\leq \lambda\left|f(a)\right| + (1-\lambda)\left|f(b)\right| \leq \left|f(a)\right| + \left|f(b)\right|.$$

Thus f is bounded above.

Since f is convex on $[a, b]$, it is continuous on (a, b) and hence f is continuous on $[c, d]$ for every subinterval $[c, d]$ with $a < c < d < b$. Therefore f is bounded on every interval $[c, d]$ where $a < c < d < b$. The only way that f could fail to be bounded on $[a, b]$ would be for either $f(a+)$ or $f(b-)$ to be $-\infty$. Assume without loss of generality that $f(a+) = -\infty$.

Then there exists a sequence of points $\{x_n\}$ such that $x_n \downarrow a$ and such that $f(x_n) \longrightarrow -\infty$. Let $y \in (a, b)$ be arbitrary. For sufficiently large n, $a < x_n < y < b$, so by Theorem 1.2,

$$\frac{f(y)-f(x_n)}{y-x_n} \leq \frac{f(b)-f(y)}{y-b}$$

Let $n \rightarrow \infty$ on the left-hand side. Since $f(x_n) \longrightarrow -\infty$, the left-hand side $\longrightarrow +\infty$, a contradiction, since the right-hand side is finite.

Remark: This procedure can be used to show that if f is convex on <u>any</u> interval, then it is bounded <u>below</u> on that interval.

By Theorem 2.1, if f is m-convex and bounded on its domain then f is convex.

<u>Theorem 3.3</u>: Suppose f is differentiable. Then f is convex if, and only if, f' is a non-decreasing function.

<u>Proof; Part 1</u>: Suppose that f' is non-decreasing.

Take two arbitrary points x, y in the domain of f and without loss of generality, assume $x < y$. Let $z = \frac{1}{2}(x+y)$.

By the Mean Value Theorem:

$f(y) - f(z) = (y-z)f'(c_1)$ where $z < c_1 < y$, and $f(z) - f(x) = (z-x)f(c_2)$ where $x < c_2 < z$. Since $c_1 > c_2$ and $y - z = z - x$, by the hypothesis $(y-z)f'(c_1) \geq (z-x)f'(c_2)$ and $f(y) - f(z) \geq f(z) - f(x)$. Simplifying gives $f(z) \leq \frac{1}{2}(f(x) + f(y))$. Since $z = \frac{1}{2}(x+y)$ then $f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y))$ which means that f is midpoint convex. f differentiable implies that f is continuous and thus bounded since the domain of f is closed and bounded. Therefore by Theorem 2.1, f is convex.

<u>Proof; Part 2</u>: Suppose that f is convex. Choose two arbitrary points x, y in the domain of f. Without loss of generality assume $x < y$. Take $h \neq 0$ and small. By the convexity hypothesis and Theorem 1.4 we have

$$\frac{f(x+h)-f(x)}{h} \leq \frac{f(y+h)-f(y)}{h}$$

By hypothesis f' exists, therefore, the limits of left and right terms above exist as $h \to 0$ and

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \leq \lim_{h \to 0} \frac{f(y+h)-f(y)}{h}$$

I.e., $f'(x) \leq f'(y)$.

Theorem 3.4: If f'' exists, then f is convex if, and only if $f'' \geq 0$.

Proof: Let $g = f'$. Since f'' exists then g is differentiable. By the Mean Value Theorem, $g' \geq 0$ if, and only if g is non-decreasing, so that the theorem follows from 3.4.

4. Conclusion

Convex functions defined on a closed bounded interval have been defined. Several properties and conditions of convexity for such functions have been established either by theorem or example. Examples of such properties and conditions are: (1) Convexity on an open interval implies continuity but does not imply boundedness. (2) Convexity on a closed bounded interval implies boundedness, but does not imply continuity.

The investigation of m-convex functions showed that a conditional hypothesis boundedness on a non-trivial closed bounded subinterval of the domain of the function would suffice to ensure that the function is convex.

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APPENDIX I

PROPERTIES OF SPECIAL FIELD M

M denotes the smallest field of real numbers containing m. I.e., M is the set of all numbers expressible in the form

$$\frac{a_0 m^r + a_1 m^{r-1} + \dots + a_r}{b_0 m^s + b_1 m^{s-1} + \dots + b_s} \ .$$

$0 < m < 1$; r, s are integers and non-negative; a_i, b_i are integers; and the denominator $\neq 0$.

The following are examples of contents of M based on m as a rational number, m as an irrational algebraic number, and m as a transcendental number:

(1) Suppose m is rational. Then M is the set of all rational numbers.

(2) Suppose m is irrational, say $m = \frac{\sqrt{2}}{2}$. Then M is the set of all numbers of the form $r_1 + r_2\sqrt{2}$ where r_1, r_2 are rational..

(3) Suppose m is transcendental, say $m = \frac{e}{3}$. Then M is the set of all rationals plus all the functional values $\frac{f(e)}{g(e)}$ where f and g are polynomials with rational coefficients and $g \neq 0$

M has the following properties:

(a) M is countable

<u>Proof</u>: Zehna and Johnson in [6] prove that algebraic numbers are countable. A simple extension shows that there are only denumerably many rational functions $r(x) = f(x)/g(x)$ where f and g are polynomials with integer coefficients.

(b) M is everywhere dense

Proof: The set M contains the set of all rationals (plus other numbers if m is irrational). The set of rationals is everywhere dense in the reals, thus M is everywhere dense.

(c) The basis Y is not countable.

Proof: R is a vector space over M, and Y is a maximal set of reals which is linearly independent over M. Suppose that Y is countable. Each $a \in R$ is a linear combination of elements in Y. There are only countably many elements of Y and hence countably many coefficients; this means there are only countably many combinations. This implies the reals are countable, a contradiction. Therefore, Y is not countable.

(d) If $y^* \in (Y-M)$, then $My^* = \left\{ \mu y^* \mid \mu \in M \right\}$ is everywhere dense in R.

Proof: Suppose, to the contrary, that My^* is not dense. Select an arbitrary $x \in R$ such that x is not in My^*.

For some $y^* \in (Y-M)$ there exists μ_1 and μ_2 such that x is in $[\mu_1 y^*, \mu_2 y^*]$. Now $\frac{\mu_1 + \mu_2}{2} = \mu_3$, is in M so that x is either in $[\mu_1 y^*, \mu_3 y^*]$ or in $[\mu_3 y^*, \mu_2 y^*]$. I.e., x is in an interval half the length of the previous interval. Continuing this process we can find a sequence $\left\{ \mu_n y^* \right\}$ such that $\mu_n y^* \longrightarrow x$, so that x is in the closure of My^*. Therefore, My^* is everywhere dense in R.

(e) Y is linearly independent over M.

Proof: By definition, a basis is linearly independent.

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